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NUMERICAL INVESTIGATION OF NONLINEAR VIBRATIONS OF VISCOELASTIC PLATES AND CYLINDRICAL PANELS IN A GAS FLOW

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Nonlinear vibrations of viscoelastic elements of aviation structures are studied. A method and an algorithm for the numerical solution of integrodifferential equations are proposed. The critical velocity of the flow past viscoelastic plates is determined.

Key words: *Nonlinear vibrations, viscoelasticity, cylindrical panels.*

In the present paper, the effect of the viscoelastic properties of structural materials on the nonlinear vibrations of cylindrical panels in a gas flow is studied. Use is made of the nonlinear equations of Marguerre's shallow thin shell theory [1–3], from which von Kármán's equations [4] are derived as a particular case.

1. Formulation of the Problem and Method of Solution. We consider a viscoelastic shallow shell of rectangular planform in a supersonic gas flow with velocity V along the generatrices.

Taking into account the viscoelastic properties of the structural material, we write Marguerre's equations for the displacements u, v, and w in Cartesian coordinates:

$$
\begin{gathered}
(1-R^*)\Big(\frac{\partial^2 u}{\partial x^2}+\frac{1-\mu}{2}\,\frac{\partial^2 u}{\partial y^2}+\frac{1+\mu}{2}\,\frac{\partial^2 v}{\partial x\,\partial y}+L_1(w)\Big)-\rho\,\frac{1-\mu^2}{E}\,\frac{\partial^2 u}{\partial t^2}=0,\\
(1-R^*)\Big(\frac{\partial^2 v}{\partial y^2}+\frac{1-\mu}{2}\,\frac{\partial^2 v}{\partial x^2}+\frac{1+\mu}{2}\,\frac{\partial^2 u}{\partial x\,\partial y}+L_2(w)\Big)-\rho\,\frac{1-\mu^2}{E}\,\frac{\partial^2 v}{\partial t^2}=0,\\
D(1-R^*)\nabla^4 w+L_3^*(u,v,w)+\rho h\,\frac{\partial^2 w}{\partial t^2}=q.
\end{gathered}
\tag{1}
$$

Here

$$
L_1(w)=-(\varkappa_x+\mu\varkappa_y)\,\frac{\partial w}{\partial x}+\frac{\partial w}{\partial x}\,\frac{\partial^2 w}{\partial x^2}+\frac{1+\mu}{2}\,\frac{\partial w}{\partial y}\,\frac{\partial^2 w}{\partial x\,\partial y}+\frac{1-\mu}{2}\,\frac{\partial w}{\partial x}\,\frac{\partial^2 w}{\partial y^2},
$$

$$
L_2(w)=-(\mu\varkappa_x+\varkappa_y)\,\frac{\partial w}{\partial y}+\frac{\partial w}{\partial y}\,\frac{\partial^2 w}{\partial y^2}+\frac{1+\mu}{2}\,\frac{\partial w}{\partial x}\,\frac{\partial^2 w}{\partial x\,\partial y}+\frac{1-\mu}{2}\,\frac{\partial w}{\partial y}\,\frac{\partial^2 w}{\partial x^2},
$$

$$
\begin{aligned}
L_3^*(u,v,w)={}&(1-R^*)\,\frac{Eh}{1-\mu^2}\Big[-(\varkappa_x+\mu\varkappa_y)\,\frac{\partial u}{\partial x}-(\mu\varkappa_x+\varkappa_y)\,\frac{\partial v}{\partial y}\\
&+(\varkappa_x^2+\varkappa_y^2+2\mu\varkappa_x\varkappa_y)w-\frac{\varkappa_x+\mu\varkappa_y}{2}\Big(\frac{\partial w}{\partial x}\Big)^2-\frac{\varkappa_x+\mu\varkappa_y}{2}\Big(\frac{\partial w}{\partial y}\Big)^2\Big]\\
&-\frac{Eh}{1-\mu^2}\,\frac{\partial}{\partial x}\Big[\frac{\partial w}{\partial x}\,(1-R^*)\Big(\frac{\partial u}{\partial x}+\mu\,\frac{\partial v}{\partial y}-(\varkappa_x+\mu\varkappa_y)w\Big)+\frac{1-\mu}{2}\,\frac{\partial w}{\partial y}\,(1-R^*)\Big(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\Big)\Big]\\
&-\frac{Eh}{1-\mu^2}\,\frac{\partial}{\partial y}\Big[\frac{\partial w}{\partial y}\,(1-R^*)\Big(\mu\,\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}-(\mu\varkappa_x+\varkappa_y)w\Big)+\frac{1-\mu}{2}\,\frac{\partial w}{\partial x}\,(1-R^*)\Big(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\Big)\Big],
\end{aligned}
$$

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$\varkappa_x$ and $\varkappa_y$ are the principal curvatures of the shell surface, D is the flexural rigidity of the shell, μ, E, and ρ are the Poisson's ratio, elastic modulus, and density of the material, respectively, h is the shell thickness, R^* is the integral operator: $R^*\varphi(t) = \int_0^t R(t-\tau)\varphi(\tau)\,d\tau$, where $R(t-\tau)$ is the relaxation kernel; the aerodynamic pressure q acting on the shell is determined according to the Il'yushin theory [5].

The boundary conditions are as follows:

$$x = 0,\ x = a{:}\qquad w = 0,\quad v = 0,\quad N_x = 0,\quad M_x = 0,$$
$$y = 0,\ y = b{:}\qquad w = 0,\quad u = 0,\quad N_y = 0,\quad M_y = 0.$$

Bending of the middle surface induces normal and shear stresses:

$$N_x = \frac{Eh}{1-\mu^2}(1-R^*)(\varepsilon_x + \mu\varepsilon_y)\quad (x \rightleftarrows y),\qquad N_{xy} = \frac{Eh}{2(1+\mu)}(1-R^*)\varepsilon_{xy}.$$

Here ε_x, ε_y, and ε_{xy} are the finite-strain components given by

$$\varepsilon_x = \frac{\partial u}{\partial x} - \varkappa_x w + \frac{1}{2}\Big(\frac{\partial w}{\partial x}\Big)^2,\quad \varepsilon_y = \frac{\partial v}{\partial y} - \varkappa_y w + \frac{1}{2}\Big(\frac{\partial w}{\partial y}\Big)^2,\quad \varepsilon_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\,\frac{\partial w}{\partial y}.$$

The moments M_x, M_y, and M_{xy} are expressed in terms of the deflection w:

$$M_x = -D(1-R^*)\Big(\frac{\partial^2 w}{\partial x^2} + \mu\,\frac{\partial^2 w}{\partial y^2}\Big),\qquad M_y = -D(1-R^*)\Big(\frac{\partial^2 w}{\partial y^2} + \mu\,\frac{\partial^2 w}{\partial x^2}\Big),$$
$$M_{xy} = D(1-\mu)(1-R^*)\,\frac{\partial^2 w}{\partial x\,\partial y}.$$

We seek an approximate solution of system (1) in the form

$$\begin{aligned}
u(x,y,t) &= \sum_{n=1}^{N}\sum_{m=1}^{M} u_{nm}(t)\cos\frac{n\pi x}{a}\sin\frac{m\pi y}{b},\\
v(x,y,t) &= \sum_{n=1}^{N}\sum_{m=1}^{M} v_{nm}(t)\sin\frac{n\pi x}{a}\cos\frac{m\pi y}{b},\\
w(x,y,t) &= \sum_{n=1}^{N}\sum_{m=1}^{M} w_{nm}(t)\sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b}.
\end{aligned}\tag{2}$$

Substituting (2) into system (1) and using the Bubnov–Galerkin procedure, we obtain the following system of integrodifferential equations in the dimensionless variables x/a, y/b, u/h, v/h, w/h, and $V_\infty t/a$ (with the previous notation):

$$\begin{aligned}
&\ddot{u}_{kl} + (1-R^*)M_E\pi^2\Big(\alpha_{kl}u_{kl} + g_{kl}v_{kl} + \omega_k w_{kl} + \frac{k_g}{\pi\lambda_1}\sum_{n,i=1}^{N}\sum_{m,r=1}^{M} D_{klnmir}w_{nm}w_{ir}\Big) = 0,\\
&\ddot{v}_{kl} + (1-R^*)M_E\pi^2\Big(g_{kl}u_{kl} + \beta_{kl}v_{kl} + \lambda\omega_l w_{kl} + \frac{k_g\lambda}{\pi\lambda_1}\sum_{n,i=1}^{N}\sum_{m,r=1}^{M} E_{klnmir}w_{nm}w_{ir}\Big) = 0,\\
&\ddot{w}_{kl} + M_\lambda\dot{w}_{kl} + (1-R^*)\Omega\Big(d_k u_{kl} + s_1 v_{kl} + \omega_{kl}w_{kl} - k_g\sum_{n,i=1}^{N}\sum_{m,r=1}^{M} p_{klnmir}w_{nm}w_{ir}\Big)\\
&\quad - k_g\sum_{n,i=1}^{N}\sum_{m,r=1}^{M} w_{nm}(1-R^*)\{A_{klnmir}u_{ir} + B_{klnmir}v_{ir} + C_{klnmir}w_{ir}\}\Omega\\
&\quad + \chi M_p\Big(2\lambda_1 M^*\sum_{n=1}^{N}\gamma_{nk}w_{nl} + \frac{\chi+1}{4}(M^*)^2 k_a\sum_{n,i=1}^{N}\sum_{m,r=1}^{M}\Gamma_{klnmir}w_{nm}w_{ir}\Big) = 0,\\
&k = \overline{1,N},\qquad l = \overline{1,M}.
\end{aligned}\tag{3}$$

Here

$$\Omega = \frac{M_E}{1-\mu^2}, \qquad \alpha_{kl} = \frac{k^2}{1-\mu^2} + \frac{l^2\lambda^2}{2(1+\mu)}, \qquad \beta_{kl} = \frac{k^2}{2(1+\mu)} + \frac{\lambda^2 l^2}{1-\mu^2},$$

$$\omega_k = (\varkappa_x + \mu\varkappa_y)\,\frac{k\beta_1\lambda_1}{\pi(1-\mu^2)}, \qquad g_{kl} = \frac{kl\lambda}{2(1-\mu)}, \qquad d_k = (\varkappa_x + \mu\varkappa_y)k\pi\lambda_1\beta_1,$$

$$\omega_{kl} = \frac{\pi^4}{12\lambda_1^2}\,(k^2 + l^2\lambda^2)^2 + (\varkappa_x^2 + \varkappa_y^2 + 2\mu\varkappa_x\varkappa_y)\lambda_1^2(\beta_1)^2,$$

$$\omega_l = (\mu\varkappa_x + \varkappa_y)\,\frac{l\beta_1\lambda_1}{\pi(1-\mu^2)}, \qquad M_E = \frac{E}{\rho V_\infty^2}, \qquad M_p = \frac{p_\infty}{\rho V_\infty^2}, \qquad \lambda_1 = \frac{a}{h}, \qquad \beta_1 = \frac{h}{R},$$

$$D_{klnmir} = \frac{ni^2}{1-\mu^2}\,\Delta_{1klnmir} + \frac{nr^2\lambda^2}{2(1+\mu)}\,\Delta_{1klnmir} - \frac{imr\lambda^2}{2(1-\mu)}\,\Delta_{2klnmir}, \qquad M_\lambda = \frac{\chi\lambda_1}{\lambda}\,M_p,$$

$$E_{klnmir} = \frac{mr^2\lambda^2}{1-\mu^2}\,\Delta_{3klnmir} + \frac{mi^2}{2(1+\mu)}\,\Delta_{3klnmir} - \frac{nir}{2(1-\mu)}\,\Delta_{4klnmir}, \qquad M^* = \frac{V}{V_\infty},$$

$$s_1 = (\mu\varkappa_x + \varkappa_y)l\pi\lambda^2\beta_1, \quad p_{klnmir} = (\varkappa_x + \mu\varkappa_y)\beta_1 n^2\Delta_{5klnmir}/2 + (\varkappa_y + \mu\varkappa_x)\beta_1\lambda m^2\Delta_{6klnmir}/2,$$

$$A_{klnmir} = (\pi/\lambda_1)[(n^2 i/\lambda + m^2 i\lambda\mu)\Delta_{7klnmir} + 2(1-\mu)nmr\lambda\Delta_{8klnmir}$$
$$- n(i^2/\lambda + r^2\lambda)\Delta_{5klnmir} - (1-\mu)mir\lambda\Delta_{6klnmir}],$$

$$B_{klnmir} = (\pi/\lambda_1)[(n^2 r\mu + m^2 r\lambda)\Delta_{7klnmir} + 2(1-\mu)nmi\Delta_{8klnmir}$$
$$- nir(1+\mu)\Delta_{5klnmir} - m(i^2 + \lambda^2 r^2)\Delta_{6klnmir}],$$

$$C_{klnmir} = (n\beta_1/\lambda)(\varkappa_x + \mu\varkappa_y)(n\Delta_{7klnmir} - i\Delta_{5klnmir}) + m\lambda\beta_1(\varkappa_x\mu + \varkappa_y)(m\Delta_{7klnmir} - r\Delta_{6klnmir}),$$

$$\Gamma_{klnmir} = ni(\gamma_{k+n+i} - \gamma_{n-k+i} - \gamma_{n-k-i} + \gamma_{k+n-i})(\gamma_{m-r+l} - \gamma_{m-r-l} - \gamma_{m+r+l} + \gamma_{m+r-l}),$$

$$\Delta_{1klnmir} = \gamma_{1kni}\gamma_{3lmr}, \qquad \Delta_{2klnmir} = \gamma_{2kni}\gamma_{3lmr}, \qquad \Delta_{3klnmir} = \gamma_{3kni}\gamma_{1lmr},$$

$$\Delta_{4klnmir} = \gamma_{4kni}\gamma_{2lmr}, \qquad \Delta_{5klnmir} = \gamma_{4kni}\gamma_{3lmr}, \qquad \Delta_{6klnmir} = \gamma_{3kni}\gamma_{4lmr},$$

$$\Delta_{7klnmir} = \gamma_{3kni}\gamma_{3lmr}, \qquad \Delta_{8klnmir} = \gamma_{4kni}\gamma_{4lmr},$$

$$\gamma_{1kni} = \gamma_{k+n+i} - \gamma_{k+n-i} - \gamma_{k-n-i} + \gamma_{k-n+i}, \quad \gamma_{2kni} = \gamma_{k+n+i} - \gamma_{k-n+i} - \gamma_{k-n-i} + \gamma_{k+n-i},$$

$$\gamma_{3kni} = \gamma_{k-n+i} + \gamma_{k+n-i} - \gamma_{k-n-i} - \gamma_{k+n+i}, \quad \gamma_{4kni} = \gamma_{k-n-i} + \gamma_{k+n+i} + \gamma_{k-n+i} + \gamma_{k+n-i},$$

$$\gamma_s = \begin{cases} 0, & s = 0 \text{ or } s = 2, 4, 6, \ldots, \\ 1/s, & s = 1, 3, 5, \ldots, \end{cases}$$

the quantities k_g and k_a are the geometric and aerodynamic nonlinearity parameters.

2. Numerical Results. To solve the nonlinear vibration problems of viscoelastic plates and nonshallow shells governed by the system (3), we employ the numerical method [6] based on quadrature formulas.

System (3) can be written in integral form by integrating twice over t. Then, setting $t = t_i$ and $t_i = i\Delta t$ ($i = 1, 2, \ldots$; $\Delta t = \text{const}$) and replacing the integrals by the trapezoidal quadrature formulas for calculating $u_{ikl} = u_{kl}(t_i)$, $v_{ikl} = v_{kl}(t_i)$, and $w_{ikl} = w_{kl}(t_i)$, we obtain recursive formulas, which are cumbersome and not given here. The calculations were performed for the Koltunov–Rzhanitsyn kernel $R(t) = A\exp(-\beta t)t^{\alpha-1}$, $0 < \alpha < 1$.

The algorithm proposed was implemented in an application software package in Delphi.

The calculation results are given in Table 1 and Figs. 1–3.

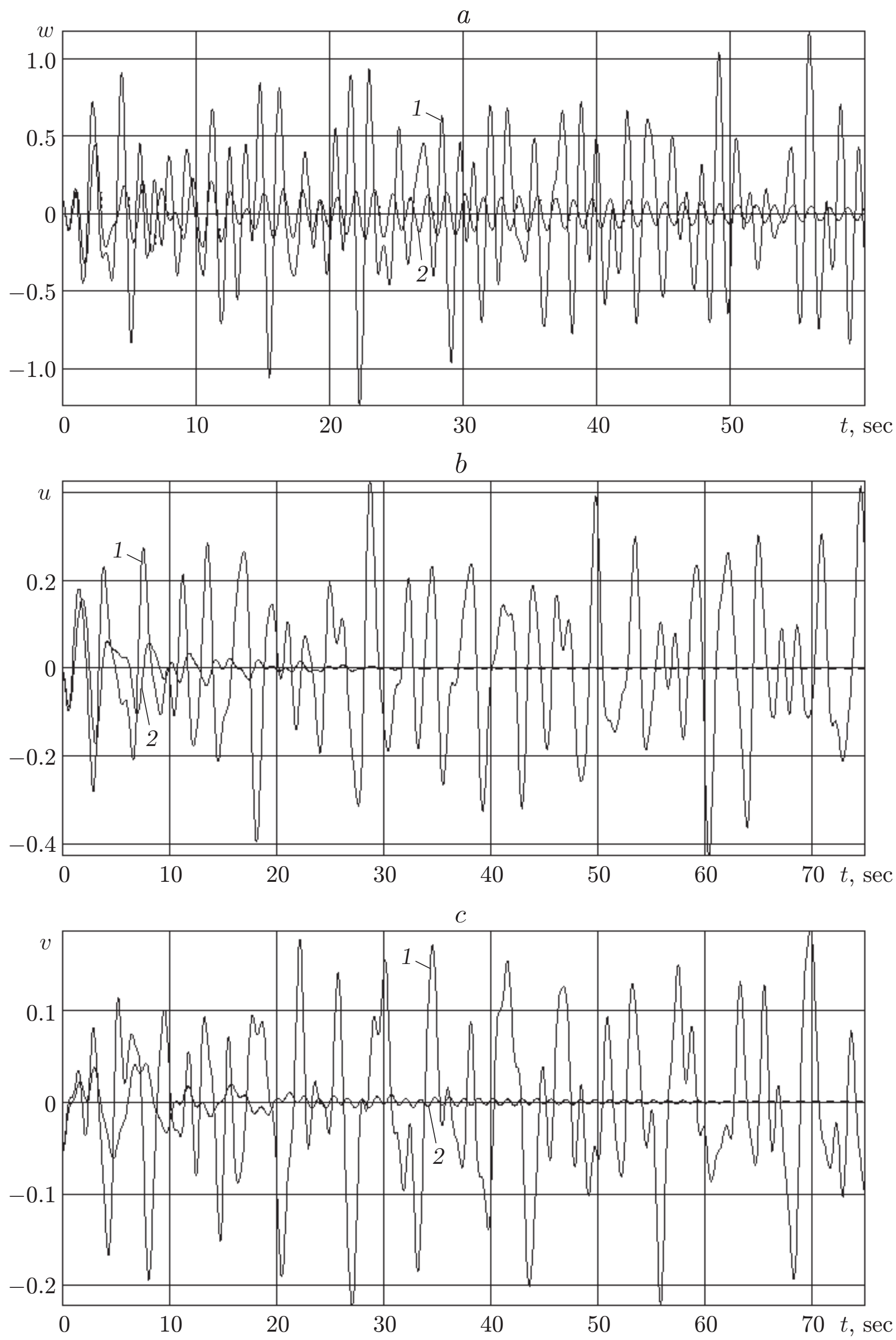

Fig. 1. Time dependence of the displacements w, u, and v of a cylindrical panel for various values of the viscosity parameter: $A = 0$ (1) and 0.1 (2); $\alpha = 0.25$, $\beta = 0.05$, $\beta_1 = 0.05$, $\lambda = 1.5$, $\lambda_1 = 75$, $\varkappa_x = 0$, $\varkappa_y = 1$, and $V = 527$ m/sec.

TABLE 1

Dependence of the Flow Critical Velocity
on the Physicomechanical and Geometric Parameters of the Plate

A	α	β	a/h	λ	$V_{\rm cr}$
0 0.005 0.01 0.10	0.25	0.05	200	1.0	750 602 523 415
0.01	0.10 0.40 0.60	0.05	200	1.0	412 528 563
0.01	0.25	0.10 0.01	200	1.0	520 525
0.01	0.25	0.05	150 180 220	1.0	830 616 410
0.01	0.25	0.05	200	1.8 2.2 2.5	552 605 653

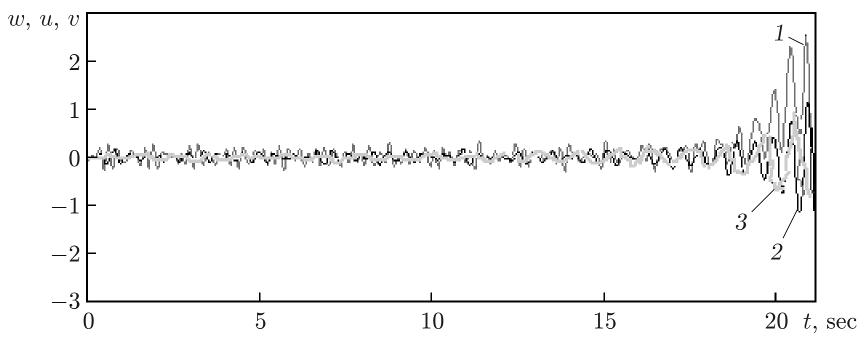

Fig. 2. Time dependences of the displacements w (curve 1), u (curve 2), and v (curve 3) of a viscoelastic plate ($A = 0.001$) for velocities higher than the critical velocity.

The flow critical velocity $V_{\rm cr}$ is defined as the velocity at which the structure performs continuous harmonic vibrations with increasing amplitude. For $V > V_{\rm cr}$, vibrations with rapidly increasing amplitudes occur, which can lead to failure of the structure. If $V < V_{\rm cr}$, the flow velocity is lower than the critical velocity and the vibrations of the viscoelastic plate damp out.

The effect of the viscoelastic properties of the plate material on the flow critical velocity was studied. The calculation results given in Table 1 and Figs. 1–3 show that the solutions of the elastic ($A = 0$) and viscoelastic ($A > 0$) problems differ substantially. For example, the flow critical velocity decreases by 44.7% if the parameter A increases from zero to 0.1.

Figure 1 shows the displacements w, u, and v of the cylindrical panel versus time for various values of the parameter A. One can see that the amplitude and frequency of the vibrations decrease as the parameter A increases.

Below, results illustrating the effect of the singularity parameter α on the flow critical velocity are given. The velocity increases with increasing α. For example, the critical velocities for $\alpha = 0.1$ and $\alpha = 0.6$ differ by 36.7%.

Figure 2 shows the displacements w, u, and v of a viscoelastic plate versus time for velocities exceeding the critical value. The vibration amplitude increases rapidly with time and flutter of the plate occurs.

From Table 1 it follows that the effect of the attenuation parameter β of the heredity kernel on the flow velocity is negligible compared to that of the viscosity parameter A and the singularity parameter α. This finding supports the well-known statement that exponential relaxation kernels are unsuitable for describing the heredity properties of structural materials.

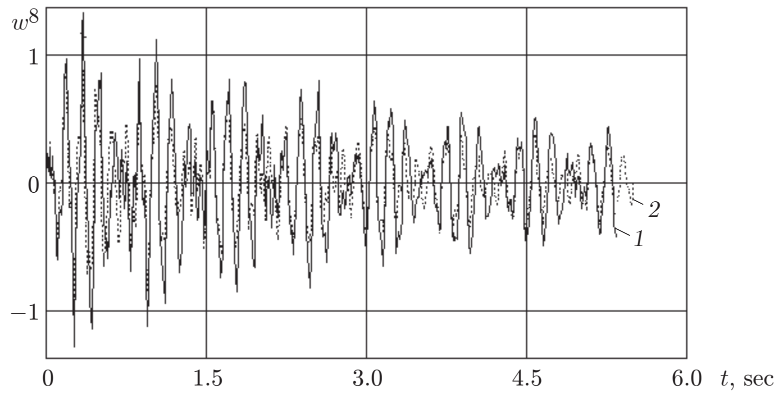

Fig. 3. Time dependence of the displacement of a cylindrical panel ignoring (curve 1) and considering (curve 2) geometric and aerodynamic nonlinearities ($A = 0.1$, $\alpha = 0.7$, $\beta = 0.05$, $\beta_1 = 0.05$, $\lambda = 6$, $\lambda_1 = 30$, $N = 3$, $\varkappa_x = 0$, $\varkappa_y = 1$, $V = 450$ m/sec): curve 1 refers to $k_g = 0$ and $k_a = 0$ and curve 2 refers to $k_g = 1$ and $k_a = 1$.

The critical flow velocity $V_{\rm cr}$ was calculated for the relative thickness of the plate $\lambda_1 = 150$, 180, and 220. The results obtained imply that the critical velocity of the flow past the viscoelastic plate decreases as the plate thickness decreases (the parameter λ_1 increases).

The effect of the aspect ratio of the plate $\lambda = a/b$ on the critical flow velocity was studied. The critical velocity increases with increasing λ. The reason is that as λ increases (for a constant value of λ_1), the relative rigidity of the system increases since the plate is diminished in the direction perpendicular to the flow.

Figure 3 shows time variations in the deflection of a cylindrical panel taking into account and ignoring geometric and aerodynamic nonlinearities. One can see that the geometric and aerodynamic nonlinearities have a significant effect on the amplitude and frequency of vibrations of the cylindrical panel; although the linear and nonlinear solutions differ only slightly in the initial stage of motion, for larger times, the amplitude of nonlinear vibrations attenuates faster than predicted by linear theory.

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